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Teaching The Wave Equation Through The Use Of Fourier's Theory

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ABSTRACT:

The teaching of differential equations is a fundamental part of the education of engineering students as a tool for use in a wide variety of applications. A remarkable role in the study of differential equations is occupied by the Wave equation due to the variety of phenomena that it allows to model. The present research aims to describe the teaching of the wave equation from the point of view of its mathematical development and the use of Fourier's theory as a solution method. In the first part of the article the mathematical framework that defines the wave equation is defined, then by the use of case studies the analytical solution of the mathematical model is developed step by step and finally by the application of the computational simulation the behavior of the solution in different circumstances is developed.

Keywords: Wave equation, Modeling education, Fourier's theory, innovation, computational simulation.

1.INTRODUCTION

The importance of the applications of differential equations for mathematical modeling shows their importance in the education of engineering students [1]. For this reason, teachers are thinking of new formative strategies for the teaching of mathematical models [2]. In this sense, modeling is a didactic tool in the teaching of differential equations for engineering careers [3].

In particular, the wave equation has become an important mathematical model for understanding a variety of natural phenomena [4]. Therefore, the study of the wave equation behavior is fundamental in the training of engineering students.
This article intends to study the wave equation to show its solution method as an important tool for teaching relevant mathematical techniques [5]. This research first of all sets out the mathematical model of the wave equation from physical processes, after which the basic concepts of Fourier's theory are defined. Subsequently, by means of two case studies, the solution of the wave equation is calculated, and its solution is simulated to define the convergence of the associated series.

2. MATHEMATICAL MODEL

The main aim of this section is to define the relevant mathematical concepts in the solution of the wave equation. Firstly, the wave equation is defined from the laws of physics, secondly, the basic elements of the Fourier theory are defined to be applied later.

2.1 Wave Equation

Consider a string of length L held at its ends on the x -axis at $x = 0$ and $x = L$. Suppose the string begins to vibrate from its initial position. Assuming that the string vibrates only in a fixed plane, let $F(x, y)$ be the function representing the transverse displacement, where time $t \geq 0$ and x is the position of the string.
In the model we assume that the rope has a constant density ρ, is perfectly elastic and the only force acting on the system is the tension force. Consider a small portion of the string between points A and B, located at x and $x + \Delta x$ as shown in Figure 2.1.

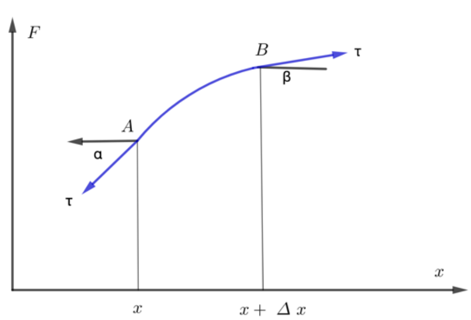

Figure 2. Forces acting on a string section

Applying Newton's second law to the vertical components of the tension force gives the equation [6]

$$-\tau \operatorname{sen}(\alpha) + \tau \operatorname{sen}(\beta) = ma, \tag{2.1}$$

where m represents the portion of the mass between x and $x + \Delta x$, and a is its acceleration. Then, $a = \frac{\partial^2 F}{\partial t^2}$ and $m = \rho \Delta x$, moreover for small angles $\operatorname{sen}(\alpha) \approx \tan(\alpha)$. The above considerations allow from the equality (2.1) to generate equation,

$$-\tau \tan(\alpha) + \tau \tan(\beta) = \rho \Delta x \frac{\partial^2 F}{\partial t^2}. \tag{2.2}$$

Since the slope of the tangent line to the graph of $F(x, y)$ is $\frac{\partial F}{\partial x}(x, t)$, then $\tan(\alpha) = \frac{\partial F}{\partial x}(x, t)$ and $\tan(\beta) = \frac{\partial F}{\partial x}(x + \Delta x, t)$. Substituting the above equalities into (2.3) gives.

$$\frac{\frac{\partial F}{\partial x}(x + \Delta x, t) - \frac{\partial F}{\partial x}(x, t)}{\Delta x} = \frac{\rho}{\tau} \frac{\partial^2 F}{\partial t^2}. \tag{2.3}$$

In equation (2.3) when $\Delta x \to 0$, the left-hand side tends to $\frac{\partial^2 F}{\partial x^2}(x, t)$, thus generating the **wave equation in one dimension** for free vibrations of a string

$$\frac{\partial^2 F}{\partial t^2}(x, t) = c^2 \frac{\partial^2 F}{\partial x^2}, \tag{2.4}$$

where $c^2 = \frac{\rho}{t}$. Note that c represents a velocity, since τ has units of $\frac{\text{length}}{\text{time}^2}$ and ρ has units of mass/length, thus c^2 has units of $\frac{\text{length}^2}{\text{time}^2}$.

2.2 Fourier Theory

The first approach to the fact of approximating a function by a series of functions occurs in the study of Differential Calculus, when considering the representation of a Taylor series around a point x_0 [7]. In this case the terms of the series are polynomial functions. There are other representations of a function in the form of an infinite series of functions, in this article we will study the Fourier series, in which the infinite series is composed of trigonometric functions.

In addition to the surprising characteristics that these series have in themselves, or their application in the study of mechanical systems or electrical systems [8], their importance lies in their usefulness to solve partial differential equations.

In this section we aim to define the Fourier series, some important characteristics and results, accompanying each aspect to facilitate their understanding.

Fourier series allow us to represent periodic functions that are important in mathematics applied to engineering.

The Fourier series of the function $f(x)$ is given by the expression [9]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{2.5}$$

The coefficients of the Fourier series (2.5) are given by the following integrals [9]:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx,\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,sen\frac{n\pi x}{L}dx \tag{2.6}$$

The following theorem gives a convergence criterion for the Fourier series (2.5) for a large class of functions.

Theorem. [9] Let $f(x)$ be a piecewise smooth function, that is, a function such that f(x) and f'(x) are continuous on the interval $[-L, L]$ (or on the interval $[-L, L]$), except perhaps at a finite number of points at which these two functions exhibit finite (avoidable or jump) discontinuities. Then the Fourier series converges to the function $f(x)$ at the points of continuity, and converges to the mean value

$$\frac{f(x+) + f(x-)}{2} \tag{2.7}$$

at each discontinuity point, where $f(x+)$ denotes the limit of f at x from the right, and $f(x-)$ denotes the limit of f at x from the left.

3. RESULTS AND DISCUSSION.

This section will show the modeling of the wave equation by using the three aspects defined in the mathematical model. Namely, the formulation of the differential equation, its subsequent solution by means of Fourier theory and finally the analysis of the convergence of the Fourier series by means of the use of software.

3.1 Vibrating string with fixed ends

The most characteristic example of the wave equation arises when the ends of the string are fixed, this is seen in the boundary conditions (BC). The initial conditions (IC) give information about the function and its partial derivative with respect to the time variable when this is zero.

Suppose the wave equation (3.1) is defined, following the model (2.4), with boundary conditions defining the fixed ends.

$$\frac{\partial^2 F}{\partial t^2} = 4\frac{\partial^2 F}{\partial x^2},\ 0 < x < \pi,\ t > 0 \qquad \text{(EDP)} \tag{3.1}$$

$$F(0,t) = 0, \quad F(\pi,t) = 0, \quad t > 0 \qquad \text{(BC)} \tag{3.2}$$

$$F(x,0) = \begin{cases} x, & si \quad 0 < x < \frac{\pi}{2} \\ \pi - x, & si \ \frac{\pi}{2} < x < \pi \end{cases}, \quad u_t(x,0) = 1 \qquad \text{(IC)} \tag{3.3}$$

To solve the wave equation (3.1) subject to the conditions (3.2) and (3.3), the separable variables method is applied [10]. This method suggests reducing the solution to the application of the theory of differential equations in one variable and Fourier's theory.

Suppose that the solution is given by the product of a function of x and another function of t, that is,

$$F(x,t) = X(x)T(t).$$

The PDE (3.1) is equivalent to the expression $X(x)T''(t) = 4X''(x)T(t)$ or with a simpler notation $XT'' = 4X''T$. When doing transposition of terms:

$\frac{X''}{X} = \frac{T''}{4T}$.

The left side of the above equation depends on x while the right side depends on t, therefore, in order to have equality the two terms must be equal to the same constant, which we will denote $-\lambda$, obtaining the expression:

$$\frac{X''}{X} = \frac{T''}{4T} = -\lambda. \tag{3.4}$$

From (3.4) we obtain the second order ODE $X'' + \lambda X = 0$.

We now analyze the boundary conditions (3.2). Since $F(x,t) = X(x)T(t)$, the first CF $F(0,t) = 0$ translates into $X(0)T(t) = 0$, and thus $X(0) = 0$. Similarly, the second CF $F(\pi,t) = 0$ implies $X(\pi)T(t) = 0$, and thus $X(\pi) = 0$.

Putting together the above steps, we must find the values of λ and the functions $X(x)$ that satisfy

$$X'' + \lambda X = 0, \quad 0 < x < \pi, \quad X(0) = 0, \quad X(\pi) = 0 \tag{3.5}$$

The ODE (3.5) is precisely a Sturm-Liouville problem [12] with eigenvalues $\lambda_n = n^2$ and the corresponding eigenfunctions, except for constant multiples, are

$X_n(x) = sen(nx)\,,\ n = 1, 2, 3, \cdots.$

Equation (3. 4) also gives rise to the following second order ODE in terms of the time variable: $T'' + 4\lambda T = 0$, and since we know that $\lambda_n = n^2$ we have $T'' + 4n^2 T = 0 \iff T'' + (2n)^2 T = 0,\ n = 1, 2, 3, \cdots$ its auxiliary equation is $r^2 + (2n)^2 = 0 \Rightarrow r = \pm 2ni$, therefore,

$T_n(t) = a_n \cos(2nt) + b_n\, sen(2nt).$

The method of separable variables tells us that the solutions sought are of the form

$F_n(x,t) = X_n(x)T_n(t) = sen(nx)\,[a_n\, cos(2nt) + b_n\, sen(2nt)]$.

The principle of superposition indicates that any finite linear combination [9] of the form

$$\sum_{n=1}^{N} F_n(x,t) = \sum_{n=1}^{N} sen(nx)\,[a_n\, cos(2nt) + b_n\, sen(2nt)] \qquad (3.6)$$

also satisfies PDE (3.1) and BC (3.2).
The function of the condition (3.3) verifies the conditions of the theorem of section 2.2, therefore it is possible to calculate the coefficients a_n y b_n that occur by means of the expressions (2.6) to generate the respective values $a_n = \frac{4}{\pi n^2} sen\frac{n\pi}{2}$. Deriving term by term the expression (4.6) with respect to t and evaluating such derivative at $t = 0$, it is deduced that

$$F(x,0) = 1 = \sum_{n=1}^{\infty} 2nb_n sen(nx) \Rightarrow 2nb_n = \frac{2}{\pi}\int_0^{\pi} sin\, nx\, dx = \frac{2[1-(-1)^n]}{n\pi} \Rightarrow$$
$$b_n = \frac{1-(-1)^n}{n^2\pi}$$

Applying (2.5), we calculate the solution of the wave equation where its respective Fourier series is

$$F(x,t) = \sum_{n=1}^{\infty} sen(nx)\left(\frac{4}{\pi n^2} sen\frac{n\pi}{2} cos(2nt) + \frac{1-(-1)^n}{n^2\pi} sen(2nt)\right) \qquad (3.7)$$

For pedagogical purposes it is possible to represent the convergence of the series (3.7) using the geogebra software [12]. Figure 3.1 shows the behavior of the series (3.7) varying $n = 5, 8{,}12$ for $t = 0.8$.

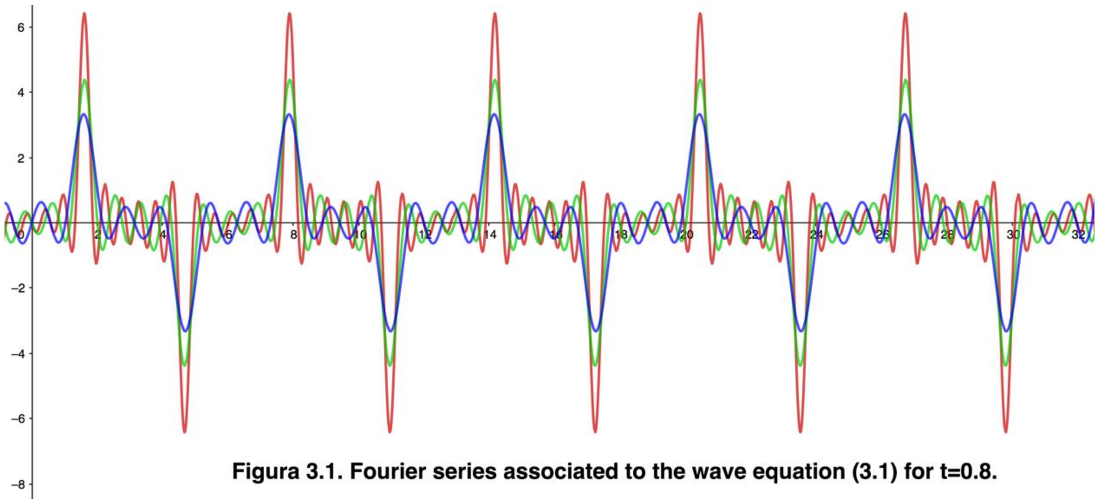

Figura 3.1. Fourier series associated to the wave equation (3.1) for t=0.8.

3.2 Vibrating string with free ends

The mathematical model that we will study in this section are very similar to those of the previous section, however, a small modification is introduced in the boundary conditions, in the previous section the fixed extremes were considered, that is to say, the solution function $F(x,t)$ is zero at the extremes of the considered interval.
Suppose the wave equation (3.1) is defined, following the model (2.4), with boundary conditions defining the fixed ends.

$$\frac{\partial^2 F}{\partial t^2} = 9\frac{\partial^2 F}{\partial x^2}, \quad 0 < x < 2, \quad t > 0 \qquad \text{(EDP)} \qquad (3.8)$$

$$F_x(0,t) = 0, \quad F_x(2,t) = 0, \quad t > 0 \qquad \text{(BC)} \qquad (3.9)$$

$$F(x,0) = 2x - x^2, \quad F_t(x,0) = 1, \quad 0 < x < 2 \qquad \text{(IC)} \qquad (3.10)$$

We again employ the method of separable variables, we assume that the solution is of the form: $F(x,t) = X(x)T(t)$. The PDE (3.8) implies the expression from this equation follows:

$$\frac{X''}{X} = \frac{T''}{9T} = -\lambda. \qquad (3.11)$$

Equation (3.8) and the BC (3.9) lead to the regular Sturm-Liouville problem

$$X'' + \lambda X = 0, \;\; 0 < x < 2, \;\; X'(0) = 0, \;\; X'(2) = 0. \qquad (3.12)$$

In a similar way to the previous section, the solution is generated for (3.12)

$$X_0(x) = 1 \text{ y } X_n(x) = cos\left(\frac{n\pi x}{2}\right), \; n = 1, 2, 3, \cdots. \qquad (3.13).$$

By using the eigenvalues in the expression (3.11) we obtain the second order ODE in terms of the time variable: $T'' + 9\lambda_n T = 0$, for $\lambda_0 = 0$ this expression is $T'' = 0 \Rightarrow T_0(t) = \frac{a_0}{2} + \frac{b_0 t}{2}$, and for the other eigenvalues we have

$$T'' + 9\lambda_n T = 0 \Leftrightarrow T'' + \left(\frac{3n\pi}{2}\right)^2 T = 0 \Rightarrow T_n(t) = a_n \cos\frac{3n\pi t}{2} + b_n \sin\frac{3n\pi t}{2}.$$

Since $u(x,t) = \sum_{n=1}^{\infty} X_n T_n$, then the solution is given by the formula:

$$F(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)\cos\left(\frac{3n\pi t}{2}\right) + \frac{b_0 t}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{2}\right)\sin\left(\frac{3n\pi t}{2}\right)$$

Evaluating the solution at $t = 0$ and using the first IC of we have

$$F(x,0) = 2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

As in the previous section, the coefficients are computed as follows:

$$a_0 = \frac{4}{3}, \qquad a_n = \frac{8[(-1)^{n+1} - 1]}{n^2\pi^2}$$

Finally, we derive $F(x,t)$ term by term with respect to t, evaluate at $t = 0$ and obtain

$$F_t(x,0) = 1 = \frac{b_0}{2} + \sum_{n=1}^{\infty} \frac{3n\pi b_n}{2}\cos\left(\frac{n\pi x}{2}\right)$$

we find another cosine Fourier series:

$$b_0 = \frac{2}{2}\int_0^2 1dx = 2, \quad \frac{3n\pi b_n}{2} = \frac{2}{2}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx = 0 \Rightarrow b_n = 0.$$

The solution is to:

$$F(x,t) = \frac{2}{3} + t + \sum_{n=1}^{\infty} \frac{8[(-1)^{n+1} - 1]}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\cos\left(\frac{3n\pi t}{2}\right) \qquad (3.9)$$

For pedagogical purposes it is possible to represent the convergence of the series (3.9) using the geogebra software. Figure 3.2 shows the behavior of the series (3.9) varying $n = 5, 8, 12$ for $t = 0.8$.

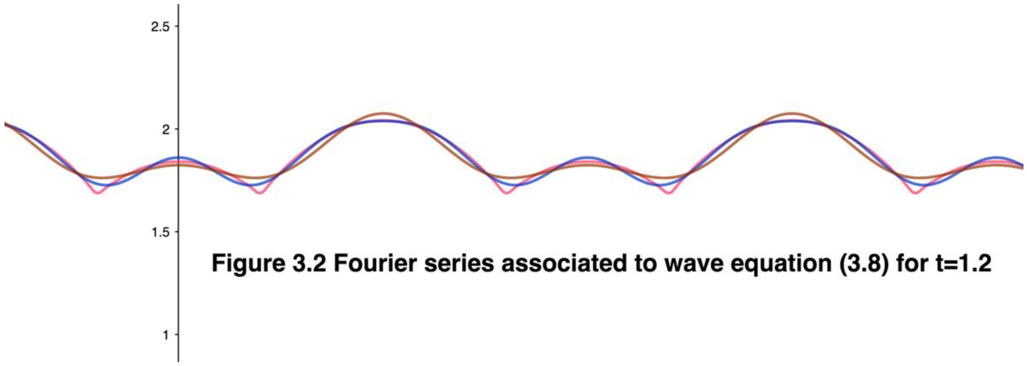

Figure 3.2 Fourier series associated to wave equation (3.8) for t=1.2

4.CONCLUSIONS

The present research derives from the laws of physics and basic concepts of differential calculus the mathematical model that represents the wave equation, this analysis strengthens the relationship between mathematical and physical concepts in the teaching of engineering students. Later on, by means of the study of two mathematical models of physical vibrations, the solution of the wave equation is calculated by means of Fourier analysis techniques, generating a series that represents the wave equation as a function of position and space. Finally, through the use of software, the convergence of the series for different values of time is described, highlighting the relevance of the use of technology in the teaching of differential equations.

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